Cramer's Rule — Linear simultaneous equations can be solved in two ways by Cramer's rule. –

i) Basic Cramer's rule ii) Alternating Cramer's Rule.

Let us suppose linear equation –

$$a_1x + b_1y = c_1 \quad --- \quad (1)$$
$$a_2x + b_2y = c_2 \quad --- \quad (2).$$

Where x and y denote two unknown variables
Now by using Cramer's rule, we obtain –

$$\frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{x}{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}.$$

For example —

1. $$3x + 4y = 5 \quad --- \quad (1).$$
$$3x - 4y = 2 \quad --- \quad (2)$$

$$\Rightarrow \frac{1}{\begin{vmatrix} 3 & 4 \\ 3 & -4 \end{vmatrix}} = \frac{x}{\begin{vmatrix} 5 & 4 \\ 2 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 5 \\ 3 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{1}{(-12-12)} = \frac{x}{(-20-8)} = \frac{y}{(6-15)}$$

$$\Rightarrow \frac{1}{-24} = \frac{x}{-28} = \frac{y}{-9}$$

$\therefore \frac{x}{-28} = \frac{1}{-24}$ and $\frac{y}{-9} = \frac{1}{-24}$

$\therefore x = \frac{-28}{-24}$ $\quad\therefore y = \frac{-9}{-24}$

$= \frac{28}{24} = \frac{7}{6}$ $\quad = \frac{9}{24}$

$= \frac{3}{8}$ Ans.

$\therefore x = \frac{7}{6}$ and $y = \frac{3}{8}$ Ans

Ex. 2. $3x + 3y - 5 = 0$

$3x - 4y = 2$

$\Rightarrow 3x + 3y = 5$

$3x - 4y = 2$

$$\Rightarrow \frac{1}{\begin{vmatrix} 3 & 3 \\ 3 & -4 \end{vmatrix}} = \frac{x}{\begin{vmatrix} 5 & 3 \\ 2 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 5 \\ 3 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{1}{(-12-9)} = \frac{x}{(-20-6)} = \frac{y}{(6-15)}$$

$$\Rightarrow \frac{1}{-21} = \frac{x}{-26} = \frac{y}{-9}.$$

$\therefore \frac{x}{-26} = \frac{1}{-21}$

$\therefore x = \frac{-26}{-21} = \frac{26}{21}$.

And $\frac{y}{-9} = \frac{1}{-21}$.

$\therefore y = \frac{-9}{-21} = \frac{9}{21} = \frac{3}{7}$

$\therefore x = \frac{26}{21}$ and $y = \frac{3}{7}$ Ans.

Ex. 3. Solve the following set of equations by Cramer's rule and matrix method.

$$2x - 3y + 5z = 11$$
$$5x + 2y - 7z = -12$$
$$-4x + 3y + z = 5$$

$$\Rightarrow \frac{1}{\begin{vmatrix} 2 & -3 & 5 \\ 5 & 2 & -7 \\ -4 & 3 & 1 \end{vmatrix}} = \frac{x}{\begin{vmatrix} 11 & -3 & 5 \\ -12 & 2 & -7 \\ 5 & 3 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 11 & 5 \\ 5 & -12 & -7 \\ -4 & 5 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -3 & 11 \\ 5 & 2 & -12 \\ -4 & 3 & 5 \end{vmatrix}}$$

Now,

$$\begin{vmatrix} 2 & -3 & 5 \\ 5 & 2 & -7 \\ -4 & 3 & 1 \end{vmatrix} = 2[2-(-21) + 3(5-28) + 5(15-(-8)]$$
$$= 2(23) + 3(-23) + 5(23)$$
$$= 46 - 69 + 115$$
$$= 161 - 69$$
$$= \underline{\underline{92}}.$$

$$\begin{vmatrix} 11 & -3 & 5 \\ -12 & 2 & -7 \\ 5 & 3 & 1 \end{vmatrix} = 11(2-(-21) + 3(-12-(-35) + 5(-36-10)$$
$$= 11(23) + 3(-12+35) + 5(-46)$$
$$= 253 + 3 \times 23 - 230$$
$$= 253 + 69 - 230$$
$$= 322 - 230 = \underline{\underline{92}}$$

$$\begin{vmatrix} 2 & 11 & 5 \\ 5 & -12 & -7 \\ -4 & 5 & 1 \end{vmatrix} \quad 2(-12+35) - 11(5-28) + 5(25-48)$$
$$= 2 \times 23 - 11 \times (23) + 5(-23)$$
$$= 46 - 253 - 115$$
$$= \cancel{161 - 253}\ 46 - 368$$
$$= \underline{\underline{-322}}$$

$$\begin{vmatrix} 2 & -3 & 11 \\ 5 & 2 & -12 \\ -4 & 3 & 5 \end{vmatrix} = 2(10-(-36)) - (-3)\times 25 - 48 + 11(15+8)$$

$$= 2(+46) + 3\times(-23) + 11(23)$$

$$= 92 - 69 + 253$$

$$= 345 - 69 = 276.$$

Now, $\dfrac{1}{92} = \dfrac{x}{92} = \dfrac{y}{-322} = \dfrac{z}{276}$

$\therefore x = \dfrac{92}{92} = 1$

$\therefore y = \dfrac{-322}{92} = 3.5$ $\boxed{\dfrac{184}{92} = 2}$ $\cancel{\dfrac{0.184}{92} = 2}$

$\therefore z = \dfrac{276}{92} = 3.$ Ans.

* — Alternative Cramer's rule —

Let us suppose two linear equations in two unknowns are as follows —

The matrix from of the above equations can be expressed as —

1. $a_1x + b_1y = c_1$ ---- ①

$a_2x + b_2y = c_2$ ---- ②

The matrix form —

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Let $\downarrow A$ $\quad \downarrow X$ $\quad \downarrow C$.

$\Rightarrow AX = C$

where $|A| \neq 0$ for existing cramer's rule

Here, $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2 \neq 0.$

using cramer's rule we get—

$$x = \frac{|\Delta_1|}{|A|} \quad \text{and} \quad y = \frac{|\Delta_2|}{|A|}$$

Here, $|\Delta_1| = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - b_1c_2$ and

$$|\Delta_2| = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - c_1a_2$$

Therefore, we get—

$$x = \frac{|\Delta_1|}{|A|} = \frac{c_1b_2 - b_1c_1}{a_1b_2 - b_1a_2} \quad \text{and}$$

$$y = \frac{|\Delta_2|}{|A|} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}$$ Ans.

2. Solve the following system of simultaneous equations by Cramer's rule.

$$5x_1 + 3x_2 = 30$$
$$6x_1 - 2x_2 = 8.$$

by using basic cramer's rule—

$$5x_1 + 3x_2 = 30 \quad \text{---} \quad (1)$$
$$6x_1 - 2x_2 = 8 \quad \text{---} \quad (2)$$

Now, by using cramer's rule, we obtain—

$$\frac{1}{\begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix}} = \frac{x_1}{\begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix}}$$

Let us suppose—

$$\frac{1}{|\Delta|} = \frac{x_1}{|\Delta_1|} = \frac{x_2}{|\Delta_2|} \quad \text{---} \quad (3)$$

Where, $|\Delta| = \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = (-10 - 18) = -28$ $\left[\because |\Delta| = -28 \neq 0\right]$

$$|\Delta_1| = \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = (-60-24) = -84.$$

$$|\Delta_2| = \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = (40-180) = -140.$$

from equation (3), we obtain –

$$\Rightarrow \frac{x_1}{|\Delta_1|} = \frac{1}{|\Delta|}$$

$$\therefore x_1 = \frac{|\Delta_1|}{|\Delta|} = \frac{\cancel{-}84}{\cancel{-}28} = \underline{\underline{3}}.$$

$$\Rightarrow \frac{x_2}{|\Delta_2|} = \frac{1}{|\Delta|}$$

$$\therefore x_2 = \frac{|\Delta_2|}{|\Delta|} = \frac{\cancel{-}140}{\cancel{-}28} = \underline{\underline{5}}.$$

Varify the value x_1 and x_2 from the equation 1 & 2.

$$5x_1 + 3x_2 = 30 \;\cdots\cdots (1)$$

$$\Rightarrow 5(3) + 3(5) = 30$$

$$\Rightarrow 15 + 15 = 30$$

$$\Rightarrow 30 = 30$$

$\therefore$ L.H.S. = R.H.S.

and.

$$6x_1 - 2x_2 = 8 \;\cdots\; (2)$$

$$\Rightarrow 6(3) - 2(5) = 8$$

$$\Rightarrow 18 - 10 = 8$$

$$\Rightarrow 8 = 8$$

$\therefore$ L.H.S. = R.H.S.

Answer. $\underline{\underline{x_1 = 3}}$ & $\underline{\underline{x_2 = 5}}$.

* Application of Cramer's Rule in equilibrium market model. –

Q1. Solve the following market model:

$D = 25 - 5P$, $S = -5 + 10P$ and $D = S$.

Solution (by Alternative Cramer's rule) —

$1.D + 0.S + 5.P = 25$ - - - ①

$0.D + 1.S - 10P = -5$ - - - ②

$1.D - 1.S + 0.P = 0$. - - - ③

The matrix form —

$$\underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -10 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} D \\ S \\ P \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 25 \\ -5 \\ 0 \end{bmatrix}}_{C}$$

$\Rightarrow AX = C$

where $|A| \neq 0$, for existing Cramer's rule. –

Here, $|A| = \begin{vmatrix} 1 & 0 & 5 \\ 0 & 1 & -10 \\ 1 & -1 & 0 \end{vmatrix}$

$= 1\{0 - (+10) - 0 + 5(0(-1) - 1.1)\}$

$= \cancel{1(10) - 0 + 5(+1)}$

$1.(-10) - 0 + 5(-1)$

$= \cancel{10 + 5}$

$= -10 - 0 - 5$

$= \underline{\underline{-15}} \neq 0$. (hence $|A| = -15 \neq 0$).

using Cramer's rule — we get.

(80)

$$\bar{D} = \frac{|\Delta_1|}{|A|}, \quad \bar{S} = \frac{|\Delta_2|}{|A|} \quad \text{and} \quad \bar{P} = \frac{|\Delta_3|}{|A|}.$$

Here, $|\Delta_1| = \begin{vmatrix} 25 & 0 & 5 \\ -5 & 1 & -10 \\ 0 & -1 & 0 \end{vmatrix}$ (D)

$= 25\{(1\times0)-(-10\times-1)\} - 0 + 5\{(-5\times-1) - 1\times0\}$

$= 25(0-10) - 0 + 5(5+0)$

$= 25\times(-10) - 0 + 5(5)$

$= -250 - 0 + 25$

$= \underline{\underline{-225}}$

$\begin{vmatrix} 1 & 25 & 5 \\ 0 & -5 & -10 \\ 1 & 0 & 0 \end{vmatrix} =$

$|\Delta_3| = \begin{vmatrix} 1 & 0 & 25 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{vmatrix}$ (P) $= 1\{1\times0 - (-5)\times(-1)\} - 0 + 25\{0\times(-1) - 1\times1\}$

$= 1(0-5) - 0 + 25(0-1)$

$= 1\times(-5) - 0 + 25(-1)$

$= -5 - 0 - 25$

$= \underline{\underline{-30}}$

$|\Delta_2| = \begin{vmatrix} 1 & 5 & 25 \\ 0 & -10 & -5 \\ 1 & 0 & 0 \end{vmatrix}$ (S) $= 1\{(-10\times0)-(-5\times0)\} - 5\{0\times0-(-5\times1)\} + 25\{0\times0 - (-10\times1)\}$

$= 1\{(0-0)\} - 5\{0-(-5)\} + 25\{-(-10)\}$

$= 1(0) - 5(5) + 25(10)$

$= 0 - 25 + 250$

$= \underline{\underline{-225}}$

Therefore we get,

$\bar{D} = \frac{|\Delta_1|}{|A|} = \frac{-225}{-15} = \underline{\underline{15}}$

$$\bar{P} = \frac{|\Delta_2|}{|A|} = \frac{-30}{-15} = \underline{\underline{2}}$$

$$\bar{S} = \frac{|\Delta_3|}{|A|} = \frac{-225}{-15} = \underline{\underline{15}}$$

$\therefore$ Equilibrium quantity :- $\bar{D} = \bar{S} = \bar{P}$.

$$\Rightarrow \cancel{\frac{15}{2}}$$

$$\Rightarrow \frac{-30}{-15} = \underline{\underline{2}}.$$

$\therefore$ Equilibrium Price $\bar{P} = \underline{\underline{2}}$ Ans.

Simple National Income Model →

Ex. 1 Given the following model-

$$Y = C + I + G$$
$$C = a + b(Y - T)$$
$$T = d + tY.$$ find Y, C, T by cramer's rule.

$$Y - C + 0 = I + G$$
$$-bY + C + bt = a$$
$$-tY + 0 + t = d.$$

$$\Rightarrow \frac{1}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{Y}{\begin{vmatrix} I+G & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix}} = \frac{C}{\begin{vmatrix} 1 & I+G & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix}} = \frac{T}{\begin{vmatrix} 1 & -1 & \cancel{I+G} \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix}}$$

Now,

$$\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} = 1(1\times1 - b\times0) - 1\{(-b)\times1 - b\times(-t)\} + 0.$$
$$= 1\times1 - 1(b + bt)$$
$$= \underline{1 - b + bt}.$$

$$\begin{vmatrix} I+G & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix} = (I+G)\times1 - (-1)(a - bd)$$
$$= I + G + 1(a - bd)$$
$$= I + G \underline{+ a - bd}.$$

$$\begin{vmatrix} 1 & I+G & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix} = 1(a\times1 - b\times d) - I+G(-b - (-bt)) + 0.$$
$$= 1(a - bd) - I+G(-b + bt).$$
$$= a - bd - \underline{I+G(-b + bt)}$$

$$\begin{vmatrix} 1 & -1 & I+G \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix} = 1(1\times d - a\times0) - (-1)(-b\times d - a\times(-t)) + (I+G)(-b\times0 - (1\times -t))$$
$$= d + (-bd + at) + \underline{(I+G)(t)}.$$

Now,
$$\frac{1}{1-b+bt} = \frac{Y}{(I+G)+a-bd} = \frac{C}{a-bd-(I+G)(bt-b)}$$

$$= \frac{T}{d+(at-bd)+(I+G)t}.$$

$$\Rightarrow \frac{Y}{(I+G)+a-bd} = \frac{1}{1-b+bt}$$

$$\therefore Y = \frac{(I+G)+a-bd}{1-b+bt}.$$

$$\Rightarrow \frac{C}{a-bd-(I+G)(bt-b)} = \frac{1}{1-b+bt}.$$

$$\therefore C = \frac{a-bd-(I+G)(bt-b)}{1-b+bt}.$$

$$\Rightarrow \frac{T}{d+(at-bd)+(I+G)t} = \frac{1}{1-b+bt}$$

$$\therefore T = \frac{d+(at-bd)+(I+G)t}{1-b+bt}$$

Ans.

Example. 2. From the following national income model find out the equilibrium national income ($\bar{Y}$) and consumption ($\bar{C}$) by using Cramer's Rule.

$$Y = C + I_0$$
$$C = 50 + 0.8Y, \quad \text{where } I_0 = \text{Rs. } 100/-$$

Solution → By using basic cramer's rule –

$$Y = C + 100.$$
$$C = 50 + 0.8y.$$

or. $Y - C = 100$ ---- (1)

$-0.8y + C = 50$ ---- (2).

using cramer's rule –

$$\Rightarrow \frac{1}{\begin{vmatrix} 1 & -1 \\ -0.8 & 1 \end{vmatrix}} = \frac{Y}{\begin{vmatrix} 100 & -1 \\ 50 & +1 \end{vmatrix}} = \frac{C}{\begin{vmatrix} 1 & 100 \\ -0.8 & 50 \end{vmatrix}}$$

Let us suppose,

$$\frac{1}{|\Delta|} = \frac{Y}{|\Delta_1|} = \frac{C}{|\Delta_2|} \quad \text{----} \ (3).$$

where, $|\Delta| = \begin{vmatrix} 1 & -1 \\ -0.8 & 1 \end{vmatrix} = (1\times 1) - (-0.8\times 1) = 1 - 0.8.$

$= \underline{0.2}$ $\left[\because |\Delta| = 0.2 \neq 0\right]$

$$|\Delta_1| = \begin{vmatrix} 100 & -1 \\ 50 & 1 \end{vmatrix} = (100\times 1) - (-1\times 50) = 100 + 50 = \underline{150}.$$

$$|\Delta_2| = \begin{vmatrix} 1 & 100 \\ -0.8 & 50 \end{vmatrix} = (1\times 50) - (100\times(-0.8))$$
$$= 50 - (-80) = 50 + 80 = \underline{130}$$

from equation (3) we obtain –

$$\frac{Y}{|\Delta_1|} = \frac{1}{|\Delta|} \quad \text{and} \quad \frac{C}{|\Delta_2|} = \frac{1}{|\Delta|}$$

$$\therefore Y = \frac{|\Delta_1|}{|\Delta|} \qquad \therefore C = \frac{|\Delta_2|}{|\Delta|}$$

$$= \frac{150}{0.2} = \underline{\underline{750}} \qquad = \frac{130}{0.2}$$

$$= \underline{650}$$

So, Equilibrium national income : $\overline{Y} = Rs.\ 750/-$

" Consumption : $\overline{C} = Rs.\ 650/-$ Ans.

Example-3 Solve the following national Income model by Cramer's rule.

$$Y = C + I + G$$
$$C = a + b(Y - t)$$
$$G = gY.$$ find Y, C, and G.

Solution :-

$$Y = C + I + G = Y - C - G = I.$$
$$C = a + b(Y - t) = -bY + C + 0 = a - bt.$$
$$G = gY \qquad = -gY + 0 + G = 0.$$

$$\Rightarrow \frac{1}{\begin{vmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -g & 0 & 1 \end{vmatrix}} = \frac{Y}{\begin{vmatrix} I & -1 & -1 \\ a-bt & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{C}{\begin{vmatrix} 1 & I & -1 \\ -b & a-bt & 0 \\ -g & 0 & 1 \end{vmatrix}} = \frac{G}{\begin{vmatrix} 1 & -1 & I \\ -b & 1 & a-bt \\ -g & 0 & 0 \end{vmatrix}}$$

Now,

$$\begin{vmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -g & 0 & 1 \end{vmatrix} = 1\{(1\times1 - 0\times0) - (-1)(-b\times1 - 0\times(-g)) + (-1)(-b\times0 - (1\times -g))\}$$
$$= 1(1) + 1(-b) - 1(+g)$$
$$= \underline{\underline{1 - b - g}}.$$

$$\begin{vmatrix} I & -1 & -1 \\ a-bt & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I\{(1\times1 - 0\times0) - (-1)((a-bt)\times1 - 0\times0) + (-1)(a-bt\times0 - 1\times0)$$
$$= I\times1 + 1(a-bt) - 1\times0.$$
$$= \underline{I + a - bt}$$

$$\begin{vmatrix} 1 & I & -1 \\ -b & a-bt & 0 \\ -g & 0 & 1 \end{vmatrix} = 1\{(a-bt\times 1)-0\times 0) - I((-b\times 1)-(0\times(-g))) + (-1)$$
$$(-b\times 0)-(a-bt\times(-g)\}$$

$$= \{1\times(a-bt) - I(-b) - 1(0-(a-bt)g\}$$
$$= a-bt + Ib + 1(a-bt)g.$$
$$= a-bt + Ib + (a-bt)g.$$
$$= a-bt + Ib + ag - btg.$$

$$\begin{vmatrix} 1 & -1 & I \\ -b & 1 & a-bt \\ -g & 0 & 0 \end{vmatrix} = 1\{(1\times 0)-(a-bt\times 0) - (-1)(-b\times 0 - \overset{(a-bt)}{\cancel{a}}\times(-g)) + I$$
$$(-b\times 0 - 1\times -g)\}$$

$$= \{(1\times 0) + 1\overset{(a-bt)g}{\cancel{(0+g)}} + I(0+g)\}$$
$$= 0 + ag - btg + Ig.$$
$$= ag - btg + Ig.$$

Now,

$$\frac{1}{1-b-g} = \frac{Y}{I+a-bt} = \frac{C}{a-bt+Ib+ag-btg} = \frac{G}{ag-btg+Ig}$$

$$\therefore Y = \frac{I+a-bt}{1-b-g}.$$

$$\therefore C = \frac{a-bt+Ib+ag-btg}{1-b-g}$$

$$\therefore G = \frac{ag-btg+Ig.}{1-b-g}$$

Ans.