* **Matrix Inversion :→** The operation of dividing one matrix directly by another does not exist in matrix theory, but equivalent of division of a unit matrix by any square matrix can be accomplished (in most cases) by a process known as "Inversion of a Matrix".

If A be a square matrix of order n and if there exist another square matrix B of the same order, such that –

$AB = BA = I$ where I denotes the identity matrix of order n, then B is called the inverse of A which is generally denoted by A^{-1}.

In ordinary algebra, if $x \times y = 1$, then $x = \frac{1}{y}$ or we say that y is inverse of x, or x is inverse of y. The product of quantity x and its inverse is one (if $\frac{1}{x}$ exist or $x \neq 0$).

Similarly, if A is a matrix and A^{-1} its inverse, then their product must be equal to the identity matrix.

$$A \times A^{-1} = I \quad (I \text{ is identity matrix}).$$

$$A^{-1} = \frac{I}{A}.$$

The methods of finding inverse matrix – There are two methods of finding inverse matrix of a non-singular matrix. i) Co-factor method. ii) Gauss elimination method.

i) Co-factor method :— if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

and that the determinant of the matrix, i.e. $|A| \neq 0$.

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}}{|A|}.$$

Example ① To find ~~|A|~~ the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Solution →

step - 1. To find $|A| = ?$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$$

$$= 1(4\times6 - 5\times5) - 2(2\times6 - 3\times5) + 3(2\times5 - 3\times4)$$

$$= 1(24-25) - 2(12-15) + 3(10-12)$$

$$= 1(-1) - 2(-3) + 3(-2)$$

$$= -1 + 6 - 6$$

$$\therefore |A| = \underline{-1 \neq 0}.$$

Since $|A| \neq 0$, hence A is non singular matrix.

step - 2.

$$\therefore \text{Co-factor of } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} & +\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(4\times6)-(5\times5) & -(2\times6)-(5\times3) & +(2\times5)-(4\times3) \\ -(2\times6)-(3\times5) & +(1\times6)-(3\times3) & -(1\times5)-(2\times3) \\ +(2\times5)-(3\times4) & -(1\times5)-(3\times2) & +(1\times4)-(2\times2) \end{bmatrix}$$

$$= \begin{bmatrix} (24-25) & -(12-15) & +(10-12) \\ -(12-15) & +(6-9) & -(5-6) \\ (10-12) & -(5-6) & +(4-4) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Step-3

$$\therefore \text{Adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Step-4. Find $A^{-1} = ?$

$$\therefore A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}}{-1}$$

$$= \begin{bmatrix} \frac{-1}{-1} & \frac{3}{-1} & \frac{-2}{-1} \\ \frac{3}{-1} & \frac{-3}{-1} & \frac{1}{-1} \\ \frac{-2}{-1} & \frac{1}{-1} & \frac{0}{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Ans.

Example-2. Find the inverse of $A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 2 & 2 \end{vmatrix} = -12$

∴ Co-factor of A means.

$$Adj(A) = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -2 \\ -5 & -7 & 11 \\ 2 & 10 & -14 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{Adj(A)}{|A|} = \frac{\begin{bmatrix} 2 & -2 & -2 \\ -5 & -7 & 11 \\ 2 & 10 & -14 \end{bmatrix}}{-12}$$

$$= \begin{bmatrix} \frac{2}{-12} & \frac{-2}{-12} & \frac{-2}{-12} \\ \frac{-5}{-12} & \frac{-7}{-12} & \frac{11}{-12} \\ \frac{2}{-12} & \frac{10}{-12} & \frac{-14}{-12} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{5}{12} & \frac{7}{12} & -\frac{11}{12} \\ -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \end{bmatrix}$$

Ans.

Solution of Simultaneous equation systems:→

There are basically two methods of finding the solution of a system of simultaneous equations. One is the direct method by using matrix inversion and other is the Cramer's rule, ~~which we will discuss in the~~.

Let us consider two linear equations in x and y,

$$a_{11}x + a_{12}y = c_1$$
$$a_{21}x + a_{22}y = c_2$$

The above equations can be converted into the following matrix form.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{C}$$

Let us suppose

$$AX = C.$$
$$\therefore X = A^{-1}C \quad \text{where} \quad A^{-1} = \frac{\mathrm{Adj}(A)}{|A|}.$$

For example → 1. Solve the system of equations $x + 2y - 4 = 0$, $2x + 5y - 9 = 0$. using inverse matrix technique.

Solution → The given system is -

$$x + 2y = 4 \quad \text{---} \quad (1)$$
$$2x + 5y = 9 \quad \text{---} \quad (2).$$

Step 1. The above equation have been converted into matrix form..

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 4 \\ 9 \end{bmatrix}}_{C}.$$

Suppose $AX = C$. $\therefore X = A^{-1}C$. where $\boxed{A^{-1} = \frac{\mathrm{Adj}(A)}{|A|}}$.

$\therefore |A| = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1.$

$\therefore |A| = 1 \neq 0$. , hence A is non singular matrix.

$\therefore$ Co-factor of $A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \Rightarrow \begin{matrix} 5 & -2 \\ -2 & 1 \end{matrix}$

$= \begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix}$

$\therefore Adj(A) = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{Adj(A)}{|A|}$

$= \frac{\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

To find values of unknown variables. -

$\therefore X = A^{-1}C$

$= \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

$= \begin{bmatrix} 5 \times 4 + (-2) \times 9 \\ -2 \times 4 + 1 \times 9 \end{bmatrix}$

$= \begin{bmatrix} 20 - 18 \\ -8 + 9 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

To varify the values of x and y.

from equation (1)

$$x + 2y = 4$$

$$2 \times 2.1 = 4$$

$$\underline{\underline{4 = 4.}}$$

$\therefore$ L.H.S. = R.H.S.

from equation (2)

$$2x + 5y = 9$$

$$2 \times 2 + 5.1 = 9$$

$$4 + 5 = 9$$

$$\underline{\underline{9 = 9}}$$

$\therefore$ LHS = RHS.

Hence required solutions are $x = 2$, $y = 1$. Ans.