* **Rank of a Matrix :—** The rank of a matrix is defined as the maximum number of linearly independent rows or columns of a matrix.

The order of the largest square submatrix, thus obtained whose determinant has a non-zero value is called the rank of the matrix A. and denoted by r. The following are the necessary rules for finding r.

i) Calculate the minors of highest possible order. If it is not zero, the order of the minor is the rank.

ii) If it is zero and every other minors of same order is zero, then calculate minor of next lower order and if at least one minor is not zero, then this order will be the rank.

iii) If however, all the minors of such order are zero, then calculate minors of still next lower order and so on.

Examples — 1. i) if $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ the $r(A) = ?$

Solution – $|A| = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$

$$|A| = 8 - 3 = 5 \neq 0$$

$$\therefore r(A) = 2$$

ii) If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ the $r(A) = ?$

$|A| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$, $|A| = 0$ $\therefore r(A) = 0$.

iii) If $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ the $r(A) = ?$

$|A| = \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix}$, $|A| = 6 - 6 = 0$.

Since $|A| = 0$, hence A is singular matrix. Let us consider a first order matrix (submatrix) to find the rank of matrix A.

$A_1 = [2]$, $|A_1| = 2 \neq 0$,

$\therefore r(A) = 1$.

Example 2. i) if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $r(A) = ?$

Solution - $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$|A| = 1(1-0) - 0(0-0) - 0(0-0)$.

$= 1.(1) = 1$.

$|A| = 1 \neq 0$ $\therefore r(A) = 3$. (3 column).

ii) $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 0 & 6 \\ 1 & 2 & 2 \end{bmatrix}_{3\times3}$ then find $r(A) = ?$

$|A| = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 0 & 6 \\ 1 & 2 & 2 \end{vmatrix}$, $|A| = 2(0-12) - 5(6-6) + 4(6-0)$

$= 2(-12) - 5.(0) + 4(6)$

$= -24 - 0 + 24$

$= 0$ $\therefore r(A) = 0$

vi) Find the rank of the matrix : $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$.

Solution –

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix}$$

$$= 1(40-42) - 2(20-21) + 3(12-12).$$

$$= 1.(-2) - 2(-1) + 3.(0).$$

$$= -2 + 2 + 0.$$

$$= \underline{\underline{0}}.$$

Since $|A| = 0$, hence A is ~~not~~ singular matrix. Let us consider a square matrix (submatrix) of order 2.

$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = |A_1| = 4-4 = 0$, but.

$A_2 = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = |A_2| = 14-12 = 2 \neq 0$.

$\therefore r(A) = 2$ Ans.

◆ 1.14. RANK OF A MATRIX

Definition. The number r is called the rank of the matrix A, if

(*i*) There exists at least one non-zero minor of A of order r.

(*ii*) Every minor of order $(r+1)$ of A vanishes.

The rank r of the matrix A is denoted by $\rho(A)$.

Ex. Find the rank of matrices A, where

(*i*) $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 6 \\ 2 & 4 & 2 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ (*iii*) $[8 \quad 5 \quad 7]$

Sol. (*i*) There is no minor of order 4.

Every minor of order 3 is zero

$\therefore \rho(A) \leq 2$

Now $$\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4 \neq 0.$$

$\therefore$ There exists a non-zero minor of order 2,

$\therefore \rho(A) = 2$

(*ii*) Here $|A| = 0$, every minor of order 2 is zero, but individual elements are non-zero.

i.e. each minor of order one is non-zero.

$\therefore$ $$\rho(A) = 1$$

(*iii*) $A = [8\ 5, 7]$

There is no minor of order two and individual elements are non-zero, so $\rho(A = 1$.

1.14.1 (*a*) Elementary Transformation

The following operations on a matrix, called elementary transformation, do not change either the order or the rank of the matrix.

(*i*) The interchange of two rows (or columns)

(*ii*) Multiplication of the elements of row (or coluumn) by a non-zero scalar.

(*iii*) Addition to the elements of a row (or column) k times the corresponding elements of another row (or column), where k is a non-zero scalar.

(*b*) Symbols for Elementary Transformation

We shall use the following symbols for row transformation :

(*i*) R_{ij} for interchange of *i*th and *j*th row.

(*ii*) $R_j\, k$ for multiplication of the *i*th row by $k \neq 0$.

(*iii*) $R_i + kR_j \to R_i$ for addition to the elements of *i*th row k times the corresponding elements of *j*th row.

Similarly we use the symbols C_{ij}, $C_i\,(k)$ and $C_i + kC_j \to C_i$ for the corresponding column transformation.

(*c*) Equivalent Matrices

Two matrices A and B are called equivalent, $A \sim B$, the one can be obtained from the other by applying a finite number of elementary transformations.

Ex. Find the rank of $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 3 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 3 \end{bmatrix}$, use $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & -2 & -3 & -6 \\ 0 & -8 & -10 & -18 \end{bmatrix}, \text{ use } C_2 \to C_2 - 3C_1,\ C_3 \to C_3 - 4C_1,\ C_4 \to C_4 - 7C_1$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -2 & -3 & -6\\ 0 & -8 & -10 & -18\end{bmatrix}, \text{ use } C_2 \to C_2\left(\frac{-1}{2}\right)$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -3 & -6\\ 0 & 4 & -10 & -18\end{bmatrix}, \text{ use } R_4 \to R_4 - 4R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -3 & -6\\ 0 & 0 & 2 & 6\end{bmatrix}, \text{ use } C_3 \to C_3 + 3C_2, C_4 \to C_4 + 6C_2$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 6\end{bmatrix}, \text{ use } C_4 \to C_4 - 3C_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\end{bmatrix}$$

We can'nt have any minnor of order 4.

But minor of order 3 is $\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{vmatrix} = 2 \neq 0$

$\therefore$ Rank of A is 3 *i.e.* $\rho(A) = 3$.

EXERCISE 1.14

Find the rank of following matrices :

1. $\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$ **2.** $\begin{bmatrix}0 & 6 & 6\\ -8 & 7 & 2\\ -2 & 3 & 0\end{bmatrix}$